

Relics of spatial curvature in the primordial non-gaussianity

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Abstract. We study signatures in the Cosmic Microwave Background (CMB) induced by the presence of strong spatial curvature prior to the epoch of inflation which generated our present universe. If inflation does not last sufficiently long to drive the large-scale spatial curvature to zero, then presently observable scales may have left the horizon while spatial slices could not be approximated by a flat, Euclidean geometry. We compute corrections to the power spectrum and non-gaussianity of the CMB temperature anisotropy in this scenario. The power spectrum does not receive significant corrections and is a weak diagnostic of the presence of curvature in the initial conditions, unless its running can be determined with high accuracy. However, the bispectral non-gaussianity parameter f_{NL} receives modifications on the largest observable scales. We estimate that the maximum signal would correspond to $f_{\text{NL}} \sim 0.3$, which is out of reach for present-day microwave background experiments.

Keywords: Inflation, Cosmological perturbation theory, Physics of the early universe, Quantum theory in curved spacetime.

1. Introduction

The precise data which led to our present era of observation-driven cosmology have largely been taken from measurements of the temperature and polarization anisotropies of the Cosmic Microwave Background [1, 2]. These anisotropies were apparently sourced by large-scale fluctuations of primordial origin, which passed inside the causal horizon while the universe was dominated by a hot plasma of relativistic matter and radiation. The existence of these long wavelength fluctuations can be inferred today by studying the imprint of the plasma oscillations which they seeded and which were left intact when radiation subsequently decoupled from matter.

The origin of these superhorizon fluctuations is uncertain. With present-day observations, their properties are indistinguishable from a precisely gaussian random field. It is possible they were synthesized during an early epoch of inflation, in which case the detectability of non-gaussian features is known to be a precise discriminant between the simplest model, characterized by a single degree of freedom [3] and more complicated models, characterized by many degrees of freedom [4]. (For example, see Refs. [5, 6, 7, 8, 9, 10].) In addition to its dependence on the dynamics of the inflationary phase, the non-gaussian signal may be sensitive to the composition of the universe before and after inflation. Subsequent to the end of the inflationary era, its evolution has been explored by several authors [11, 12, 13, 14, 15, 16]. On the other hand, if inflation persisted for some number of e-folds prior to observable scales leaving the horizon then the impact of any pre-inflationary physics can be expected to be quite negligible [17].

If inflation did not last long enough to erase all memory of earlier times, what information could be expected to survive in the statistics of large-scale fluctuations? Chen *et al.* [18] studied the impact of excitations above the Minkowski vacuum[‡] and found them to have a distinctive momentum dependence. Moreover, under certain assumptions the traces of these excitations could be made large. More detailed investigations were subsequently carried out in Refs. [20, 21]. Statistical features of this sort could probe an enhancement of the correlations predicted by conventional inflationary models due to perturbative effects, but would necessarily have limited reach to detect a coherent modification of the background spatial geometry. In the conventional picture, where inflation is taken to have lasted for many e-folds prior to horizon exit of observable scales, the spatial geometry is taken to be flat to high precision. If this assumption is relaxed, however, then the background may be different. To study such a scenario one requires a new set of propagators and vertices, which are the elements out of which perturbation theory is constructed. Predictions obtained in this way could quite easily be rather different to conventional results. In the case of both perturbative or non-perturbative enhancements the effect diminishes as spatial slices inflate and isotropize, unless the modification is associated with new physics at

[‡] When promoted to de Sitter space, the choice of boundary conditions which reproduces the Minkowski vacuum on small scales is usually known as the Bunch–Davies state [19].

an energy somewhat higher than the inflationary scale.[§]

According to present ideas the pre-inflationary phase could have been characterized by strong spatial disorder, of which the simplest manifestation would be non-vanishing spatial curvature. Assuming dark energy to be non-dynamical, observation presently requires $-0.0178 < \Omega_k < 0.0066$ at 95% confidence, where Ω_k is evaluated today and the constraints are obtained by combining the WMAP 5-year dataset with baryon acoustic oscillation and supernova data [2]. As observations improve the allowed range of Ω_k can be expected to diminish, but if it remains consistent with zero it will presumably be impossible to rule out a small positive or negative value. One might be led to seek more information in the statistical properties of the large-scale perturbations, but it is already known that curvature is hard to detect in the power spectrum alone [22]. For that purpose it seems worthwhile to explore whether the situation is better for higher-order correlations. Thus, both to decide what can be learned about the pre-inflationary initial conditions, and to determine whether curvature at the level $|\Omega_k| \sim 10^{-2}$ can be probed by non-gaussian statistics, one would like to obtain the inflationary bispectrum computed with non-flat spatial slices. In this paper we focus on the case of positive spatial curvature, corresponding to S^3 spatial slices, since in that case the observational limit is more favourable, as discussed above.

How large an effect should we expect? The longest observable wavelength in the CMB contributes to its dipole, although one must be wary in extracting data at low multipoles because of scatter due to cosmic variance. Nevertheless, as a simple estimate, a calculation of the comoving distance to the CMB dipole (or $\ell = 2$ mode; cf. Ref. [23]) gives the diameter of the last scattering sphere as

$$d_{\ell=2} = 2 \int_{t_{\text{dc}}}^{t_0} \frac{dt}{e^\rho} = 2 \int_{(1+z_{\text{dc}})^{-1}}^1 \frac{dx}{x \sqrt{-1 + x^2(\Omega_m x^{-3} + \Omega_\Lambda + \Omega_r x^{-4})/|\Omega_k|}}, \quad (1)$$

where we have specialized to S^3 slices and, as above, the Ω_i are evaluated today (at time t_0). The decoupling time is denoted t_{dc} and e^ρ is the scale factor. For the purposes of making an estimate, let us assume that the present-day spatial curvature takes the WMAP5+BAO+SN central value (at 68% confidence) $\Omega_k \approx -0.0050$ [2]. The remaining cosmological parameters can be chosen equal to the best-fit concordance model where $\Omega_\Lambda = 0.721$, $\Omega_r = 8 \times 10^{-5}$ and the CMB decouples at a redshift $z_{\text{dc}} = 1100$. This gives $d_{\ell=2} \approx 0.46$. On the other hand, in these units, the radius of curvature of the spatial slices is set by the scale $r = 1$. It follows that the CMB dipole receives contributions from wavelengths which are a significant fraction of the radius of curvature. Where this is the case, we do not expect to be able to neglect spatial curvature in calculating microwave background observables.

In this paper we estimate the observational signatures of this scenario. We study the correction to the power spectrum from single-field, slow-roll inflation assuming that

[§] If there is a sensitivity to new ultra-violet physics, then quantum excitations can have correlations which are not well-described by the standard formulation. Such modifications are not inflated away, but instead become imprinted in each new mode which crosses the horizon [18, 20, 21].

inflation began while Ω_k was non-negligible, and then consider the implications for the three-point function of the curvature perturbation. Our result agrees with the known flat-slicing result [3] in the small-scale limit $k \gg 1$ (where k is the wavenumber of the mode under consideration), in agreement with the naïve expectation that curvature should play a less important role as inflation proceeds. On the other hand for small k the fluctuations can feel the curvature of the S^3 slices, leading to terms which are larger than the flat \mathbb{R}^3 prediction by a factor of $1/\epsilon$, where $\epsilon \equiv \dot{\phi}^2/2\dot{\rho}^2$ is the slow-roll parameter which measures the deviation of the universe from exact de Sitter space. This enhancement is driven by quantum interference among the fluctuations at horizon-crossing, and is maximized close to the equilateral configuration where all momenta participating in the three-point function have an approximately equal magnitude.||

The layout of this paper is as follows. In §2 we compute the action for a scalar field coupled to Einstein gravity in de Sitter space with S^3 spatial hypersurfaces. After including the effect of small perturbations in the metric, this controls how small fluctuations interact and become non-linear as the expansion of the universe draws them across the causal horizon. In order to obtain a description of the non-linearities which are observable in the CMB bispectrum, it is sufficient to expand the action to third order in the perturbations. In §3 we study corrections to the two-point function, and show that curvature plays essentially no role in determining its structure at tree level. In §4 we give the third order action and in §5 we calculate the three point function. We then discuss the non-gaussian parameter f_{NL} imprinted in this scenario and show that we have the expected limit for modes of small wavelength. Finally, we give our conclusions in §6.

Throughout this paper, we work in units where the $\hbar = c = 1$ and the reduced Planck mass is normalized to unity, giving $M_{\text{P}}^{-2} \equiv 8\pi G = 1$, where G is Newton's gravitational constant. The unperturbed background metric is

$$ds^2 = -dt^2 + e^{2\rho} \gamma_{ij} dx^i dx^j, \quad (2)$$

and γ_{ij} is the metric on the unit three-sphere. This choice means that the constant curvature k is normalized to unity, and to avoid clutter we will omit writing k explicitly.

2. Computation of the action

To study the evolution of small fluctuations around the time of horizon exit, one must couple gravity to whichever fields are responsible for driving inflation. Since the energy density in these fields dominates the energy budget of the universe by assumption, small fluctuations necessarily induce perturbations in the background metric. The non-linearity of gravity endows these perturbations with self-interactions which cannot be neglected.

|| Therefore, we do not expect that this enhancement by $1/\epsilon$ would provide an explanation for the observation of large non-gaussianity reported by Yadav & Wandelt [24].

To simplify the subsequent calculation, it is very convenient to take the perturbed metric in the Arnowitt–Deser–Misner or so-called *ADM* form [25], where

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \quad (3)$$

In this equation N and N^i are the lapse function and shift-vector, whose function is to guarantee reparametrization invariance of the metric. They are not propagating degrees of freedom but are instead removed by constraint equations coming from the Einstein action; and h_{ij} is the perturbed metric on spatial slices,

$$h_{ij} = e^{2(\rho+\zeta)}(\gamma_{ij} + \Gamma_{ij}). \quad (4)$$

The quantity ζ is known as the *curvature perturbation*. If gravitational waves are present, represented here by a transverse and traceless first-order tensor Γ_{ij} , then they should be included as corrections to the unit three-sphere metric γ_{ij} . We have implicitly adopted a spatial gauge in order to write Eq. (4). In the present paper we focus solely on the properties of scalar degrees of freedom, which can be summarized in terms of the curvature perturbation, and set $\Gamma_{ij} = 0$. This is adequate for the purpose of computing the bispectrum of ζ . In the absence of perturbations, the lapse function and shift-vector reduce to unity and zero, respectively, and otherwise can be expressed algebraically in terms of the propagating degrees of freedom.

The action is taken to be the standard Einstein–Hilbert term, together with a real scalar field ϕ , supplemented with the York–Gibbons–Hawking boundary term,

$$S = \frac{1}{2} \int dt d^3x \sqrt{h} \left\{ N (R^{(3)} - 2V - h^{ij} \partial_i \phi \partial_j \phi) + \frac{1}{N} (E_{ij} E^{ij} - E^2 + [\dot{\phi} - N^i \partial_i \phi]^2) \right\}, \quad (5)$$

where E_{ij} is a rescaled version of the extrinsic curvature, defined by $K_{ij} \equiv E_{ij}/N$ and given explicitly by

$$E_{ij} \equiv \frac{1}{2} (\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i). \quad (6)$$

The scalar E is its trace, obeying $E \equiv E^i_i$, where spatial indices are raised and lowered with the metric h_{ij} . The covariant derivative compatible with this metric is denoted ∇_i and an overdot denotes a derivative with respect to cosmic time, so that $\dot{x} \equiv dx/dt$. There are two scalar degrees of freedom in this system: the fluctuations ζ and $\delta\phi$. However, only one of these is a propagating mode; the other is a gauge degree of freedom and can be removed by a coordinate redefinition. Although a wide variety of gauge choices exist [26], two are of special interest. The first is *comoving gauge*, where the scalar perturbation is set to zero and only the curvature perturbation propagates,

$$\delta\phi = 0, \quad \text{and} \quad h_{ij} = e^{2\rho+2\zeta} \gamma_{ij}. \quad (7)$$

Late on when a mode has passed outside the horizon γ_{ij} can be effectively replaced by a flat metric on \mathbb{R}^3 since the equations of motion are negligibly affected by the curvature of the S^3 slices. In this limit the scale of the metric is arbitrary and the equations of motion possess a symmetry under dilatations of the scale factor, where $\rho \mapsto \rho + \alpha$ for any constant α . It follows that on very large scales ζ must be effectively conserved,

because it becomes indistinguishable from a shift in the background value of ρ . This conservation law has been shown to follow—independently of the theory of gravity in question [27]—from energy conservation and the absence of non-adiabatic pressure. It is therefore convenient to express the answer in terms of ζ when outside the horizon.

Unfortunately, the calculation of correlation functions directly in comoving gauge is technically complicated. It is more convenient to work in the *uniform curvature gauge*, which is defined by

$$\delta\phi = \varphi, \quad \text{and} \quad h_{ij} = e^{2\rho}\gamma_{ij}. \quad (8)$$

In this gauge it is simple to compute the correlation functions of φ , but more work is needed to make a gauge transformation into the observationally-relevant variable ζ . It is this approach which we will follow in the remainder of this paper.

2.1. Constraint equations

The action (5) does not depend on time derivatives of N or N^i , and therefore its variation with respect to these fields does not produce evolution equations but only constraints. We need to solve these constraint equations to first order[¶]. The Hamiltonian constraint is

$$R^{(3)} - 2V - \frac{1}{N^2}(E_{ij}E^{ij} - E^2) - \frac{1}{N^2}(\dot{\phi} - N^i\phi_{,i})^2 - h^{ij}\phi_{,i}\phi_{,j} = 0, \quad (9)$$

where we have adopted the convention that a partial derivative with respect to x^i is denoted with a comma, so that $\phi_{,j} \equiv \partial_j\phi$. The momentum constraint is

$$\nabla_i \left\{ \frac{1}{N}(E_j^i - \delta_j^i E) \right\} - \frac{1}{N}(\dot{\phi} - N^i\phi_{,i})\phi_{,j} = 0 \quad (10)$$

In subsequent expressions, to simplify the notation, we will use $\phi = \phi(t)$ to denote the background value of the field and reserve φ for its perturbation.

One can now specialize to the uniform curvature gauge. In this gauge, the solution to the constraints takes the form $N \equiv 1 + \delta N$, where δN obeys

$$\delta N \equiv \frac{1}{\dot{\rho}} \left(\frac{\dot{\phi}}{2} \varphi + \chi \right); \quad (11)$$

and $N_i \equiv \chi_{,i}$, where χ is an auxiliary quantity determined by

$$\chi \equiv \epsilon(\Delta + 3 - \epsilon)^{-1} \left[\frac{d}{dt} \left(-\frac{\dot{\rho}}{\dot{\phi}} \varphi \right) + \frac{e^{-2\rho}}{\dot{\phi}} \varphi \right] \quad (12)$$

and Δ is the laplacian on the unit three-sphere. We note that both δN and N^i are suppressed by the square root of a slow-roll parameter,

$$\delta N = \mathcal{O}(\sqrt{\epsilon}) \quad \text{and} \quad N^i = \mathcal{O}(\sqrt{\epsilon}), \quad (13)$$

where $\epsilon \equiv \dot{\phi}^2/2\dot{\rho}^2$ is the slow-roll parameter defined in §1. This scaling will be extremely important when we come to extract the terms of lowest order in the slow-roll approximation from the action.

[¶] As noted in Ref. [3] and given in more detail in Ref. [28], the higher order terms in N and N^i occur in the action multiplying the Hamiltonian and momentum constraints respectively, and hence they can be neglected.

2.2. The second order action

In the uniform curvature gauge, Eq. (8), the second order term in the action is

$$S_2 = \frac{1}{2} \int dt d^3x e^{3\rho} \sqrt{\gamma} \left[-V_0'' \varphi^2 - \frac{\dot{\phi}}{\dot{\rho}} V_0' \varphi^2 - \frac{2}{\dot{\rho}} V_0' \varphi \chi + \dot{\varphi}^2 - e^{-2\rho} \gamma^{ij} \varphi_{,i} \varphi_{,j} \right. \\ \left. - 2\chi \Delta \chi - \frac{\dot{\phi}^2}{\dot{\rho}} \varphi \dot{\varphi} - \frac{2\dot{\phi}}{\dot{\rho}} \chi \dot{\varphi} + \left(\frac{\dot{\phi}^2}{\dot{\rho}^2} - 6 \right) \left(\frac{\dot{\phi}^2}{4} \varphi^2 + \dot{\phi} \chi \varphi + \chi^2 \right) \right] \quad (14)$$

where χ is given by Eq. (12) and Δ is again the laplacian on the unit three-sphere. Since $V_0'' = -3e^{-2\rho} + O(\epsilon)$ and (13) holds we have at lowest order in slow roll

$$S_2 = \frac{1}{2} \int dt d^3x e^{3\rho} \sqrt{\gamma} [\dot{\varphi}^2 + 3e^{-2\rho} \varphi^2 - e^{-2\rho} \gamma^{ij} \varphi_{,i} \varphi_{,j}]. \quad (15)$$

The field equation for φ which follows from this action is

$$\frac{d}{dt} \left(e^{3\rho} \frac{d}{dt} \varphi \right) - 3e^\rho \varphi - e^\rho \Delta \varphi = 0. \quad (16)$$

As discussed below Eq. (7), the terms depending on curvature scale out of Eq. (16) in the limit $e^\rho \rightarrow \infty$ after which Eq. (16) has a time-independent solution.

3. Corrections to the power spectrum from curvature

3.1. Scalar mode functions on the three-sphere

We wish to solve Eq. (16) at lowest order in the slow-roll approximation. For this purpose it is first necessary to obtain an expression for e^ρ at lowest order. One does this by Taylor expanding the scale factor about a particular time and collecting the terms of lowest order. In principle, the Taylor expansion can be performed at any point. However, because our final answer will involve an integral which receives its largest contribution from around the time of horizon crossing, we can achieve the greatest accuracy by expanding around the time of horizon exit. This is precisely analogous to the expansion of H and other background quantities around the time of horizon exit in the standard calculation on flat spatial slices.

The background equations for the scale factor consist of the Friedmann constraint,

$$3\dot{\rho}^2 = \frac{1}{2}\dot{\phi}^2 + V_0 - 3e^{-2\rho}, \quad (17)$$

together with the Raychaudhuri equation

$$\ddot{\rho} = -\frac{1}{2}\dot{\phi}^2 + e^{-2\rho}. \quad (18)$$

In addition, the scalar field evolves according to the conventional Klein–Gordon equation

$$\ddot{\phi} + 3\dot{\rho}\dot{\phi} + V_0' = 0, \quad (19)$$

which is not corrected in the presence of curvature. At lowest order in slow roll we can write the n th order time derivative of the scale factor in terms of alternating expressions

for odd and even n ,

$$\left(\frac{d}{dt} \right)^{2n} e^\rho \Big|_{t=t_*} = e^{\rho_*} (\dot{\rho}_*^2 + e^{-2\rho_*})^n + \mathcal{O}(\epsilon); \quad (20)$$

$$\left(\frac{d}{dt} \right)^{2n+1} e^\rho \Big|_{t=t_*} = e^{\rho_*} \dot{\rho}_* (\dot{\rho}_*^2 + e^{-2\rho_*})^n + \mathcal{O}(\epsilon). \quad (21)$$

where an asterisk $*$ denotes evaluation at the time of horizon crossing for the corresponding wavenumber k . It follows that after expansion around the time $t = t_*$, the scale factor can be expressed at lowest order in the slow-roll approximation by

$$e^\rho \simeq e^{\rho_*} \cosh[(t - t_*)\sqrt{\dot{\rho}_*^2 + e^{-2\rho_*}}] + \frac{\dot{\rho}_* e^{\rho_*}}{\sqrt{\dot{\rho}_*^2 + e^{-2\rho_*}}} \sinh[(t - t_*)\sqrt{\dot{\rho}_*^2 + e^{-2\rho_*}}]. \quad (22)$$

We can see that this is the scale factor for a de Sitter spacetime with ρ and $\dot{\rho}$ matched to our slowly rolling background at $t = t_*$. In fact, it is usually more convenient to work in terms of a conformal time variable, η , defined by $\eta = \int_{t_{\text{throat}}}^t dt'/e^\rho(t')$, where t_{throat} corresponds to the throat of the de Sitter hyperboloid. In terms of the η variable, Eq. (22) can be written

$$e^\rho = \frac{1}{\dot{\rho}_* \lambda \cos \eta} + \mathcal{O}(\epsilon). \quad (23)$$

It follows that in the neighbourhood of $\eta = \eta_*$ the Hubble rate obeys

$$\dot{\rho} = \dot{\rho}_* \lambda \sin \eta + \mathcal{O}(\epsilon), \quad (24)$$

where $\lambda \equiv \sqrt{1 + e^{-2\rho_*}/\dot{\rho}_*^2}$ and $\eta \in (-\pi/2, \pi/2)$ for $t \in \mathbb{R}$. The relation between conformal and physical time is

$$\tan \frac{\eta}{2} = \tanh \frac{1}{2} \left[(t - t_*) \dot{\rho}_* \lambda + \tanh^{-1} \left(\frac{1}{\lambda} \right) \right] \quad (25)$$

Combining Eqs. (16) and (23), we find that the equation for the modes at lowest order in slow roll is

$$\varphi_k'' + 2 \tan \eta \varphi_k' + (k^2 - 4) \varphi_k = 0 \quad (26)$$

where a prime $'$ denotes a derivative with respect to η , and $-(k^2 - 1)$, with $k \in \mathbb{N} - \{0\}$, are the eigenvalues of the laplacian Δ . This is solved by the normalised positive mode [29] constructed to vanish at the south pole of the Hawking–Moss instanton [30] describing de Sitter space at horizon crossing,

$$\begin{aligned} \varphi_k^{\text{cl}} = & \frac{\dot{\rho}_* \lambda (k^2 - 3)^{1/4}}{\sqrt{2(k^2 - 4)}} \left[F \left(\begin{matrix} -1/2 + \sqrt{k^2 - 3}/2; & -1/2 - \sqrt{k^2 - 3}/2 \\ & 1/2 \end{matrix} \middle| \sin^2 \eta \right) \right. \\ & \left. + \imath \frac{(k^2 - 4)}{\sqrt{k^2 - 3}} \sin \eta F \left(\begin{matrix} \sqrt{k^2 - 3}/2; & -\sqrt{k^2 - 3}/2 \\ & 3/2 \end{matrix} \middle| \sin^2 \eta \right) \right] \end{aligned} \quad (27)$$

where F is the Gauss hypergeometric function and $\imath^2 \equiv -1$. The south pole is at $\eta = \imath\infty$. This is equivalent to the usual formulation in which the positive mode is taken to approach its Minkowski value deep inside the horizon, where it is insensitive to the curvature of spacetime.

Ultimately we wish to use these mode functions to obtain analytic estimates of the n -point expectation values of fluctuations in φ around the time of horizon crossing. However, the presence of hypergeometric functions in Eq. (27) means that it is awkward to use such expressions for this purpose. To obtain a more useful approximation, one may first re-write the equation of motion, Eq. (26), as

$$\left(\frac{\varphi_k}{\cos \eta}\right)'' = -(k^2 - 5 - 2 \tan^2 \eta) \frac{\varphi_k}{\cos \eta}. \quad (28)$$

The equation admits an “almost-WKB” solution, corresponding to

$$\varphi_k \approx \frac{\dot{\rho}_* \lambda}{\sqrt{2k}} e^{ik\eta} \cos \eta, \quad (29)$$

which becomes a good approximation for an increasingly large neighbourhood of $\eta = 0$ as $k \rightarrow \infty$. Unfortunately, we cannot conclude that Eq. (29) is sufficient to obtain an acceptable estimate of the three-point expectation value of scalar fluctuations near horizon crossing (where $\eta \rightarrow \pi/2$), because it freezes out to the wrong asymptotic value. This is a fatal deficiency. It would lead to an unreliable estimate of the magnitude of the bispectrum and therefore untrustworthy conclusions regarding the observational relevance of f_{NL} . To find an approximate form for the mode which gives a good approximation over the *entire* relevant range of η , we can use standard results relating to hypergeometric functions to find that at late times $\eta \rightarrow \pi/2$ the field freezes out to

$$\varphi_k^{\text{cl}} \rightarrow -\frac{i\dot{\rho}_* \lambda}{(k^2 - 3)^{1/4} \sqrt{2(k^2 - 4)}} e^{i\pi\sqrt{k^2 - 3}/2} \approx -\frac{i\dot{\rho}_* \lambda}{\sqrt{2k^3}} e^{i\pi k/2}, \quad (30)$$

where the approximation holds for large k . It follows that if we take φ_k^{cl} to be given by

$$\varphi_k^{\text{cl}} \approx \frac{\dot{\rho}_* \lambda}{\sqrt{2k}} \left(\cos \eta - \frac{i}{k} \right) e^{ik\eta}, \quad (31)$$

then we obtain the correct value and derivative at $\eta = \pi/2$, and indeed this gives a good estimate for all η .

In Eq. (31) and below, the label “cl” indicates that this set of mode functions form a good basis out of which we can build a quantum field when we come to quantize this theory. To do so, one introduces creation and annihilation operators a_k^\dagger and $a_{\vec{k}}$ which respectively create and destroy particles (as measured by an inertial observer deep inside the horizon) with momentum \vec{k} . The quantum field corresponding to φ can be constructed by the usual canonical procedure, leading to

$$\varphi(\eta, \vec{x}) = \sum_{\vec{k}} \left(Q_{\vec{k}}(\vec{x}) \varphi_k^{\text{cl}}(\eta) a_k^\dagger + Q_{\vec{k}}^*(\vec{x}) \varphi_k^{\text{cl}*}(\eta) a_{\vec{k}} \right) \quad (32)$$

where the S^3 harmonics $Q_{\vec{k}}$ are defined by $\Delta Q_{\vec{k}} = -(k^2 - 1)Q_{\vec{k}}$. The case $k = 1$ is the homogeneous background and the case $k = 2$ gives purely gauge modes.⁺ For

⁺ To see that the $k = 2$ modes are pure gauge, suppose φ_2 satisfies $\Delta \varphi_2 = -3\varphi_2$. It must also satisfy $\varphi_{2|ij} = -\gamma_{ij}\varphi_2$, as can be verified by checking each $k = 2$ mode, and therefore it is possible to perform a gauge transformation from the metric

$$ds^2 = -N^2 dt^2 + e^{2\rho} \gamma_{ij} (dx^i + N^i dt)(dx^j + N^j dt), \quad \varphi = \varphi_2 \quad (33)$$

the purposes of studying fluctuations neither of these modes are relevant, and we will therefore always be interested in modes of wavenumber three or larger, so that $k \geq 3$.

3.2. Appropriate coordinates

We use coordinates in which the metric on S^3 takes the form

$$ds_3^2 = d\chi^2 + \sin^2 \chi d\Omega_2^2 \quad (36)$$

where $d\Omega_2^2$ is the usual metric on S^2 . These coordinates are convenient because if we take our position to be given by $\chi = 0$ then the CMB can be thought of as located on the copy of S^2 at $\chi = \chi_L$, where the subscript L denotes the time of last scattering. In this form the harmonics $Q_{\vec{k}}$ are given by

$$Q_{\vec{k}} \equiv \Pi_k^l(\chi) Y_l^m(\theta, \phi), \quad (37)$$

where $\vec{k} \equiv (k, l, m)$, the Y_l^m are the usual normalised harmonics on S^2 and

$$\Pi_k^l(\chi) = \sqrt{\frac{2(k-1-l)!k}{\pi(k+l)!}} (2^l l! \sin^l \chi) C_{k-1-l}^{l+1}(\cos \chi). \quad (38)$$

In Eq. (38), C_{k-1-l}^{l+1} is a Gegenbauer polynomial and $k-1 \geq l \geq |m|$. The list $\vec{k} = (k, l, m)$ labels the quantum numbers associated with each harmonic, with k labelling the “radial” χ -harmonic and (l, m) the usual harmonic labels on S^2 associated with the spherical harmonics Y_l^m . Under the exchange $m \mapsto -m$, the harmonics $Q_{\vec{k}}$ have the property* that $Q_{\vec{k}}^* = (-1)^m Q_{-\vec{k}}$. Therefore we may write the field (32) in the convenient form

$$\varphi(\eta, \vec{x}) = \sum_{\vec{k}} Q_{\vec{k}}(\vec{x}) \left[\varphi_k^{\text{cl}}(\eta) a_k^\dagger + (-1)^m \varphi_k^{\text{cl}*}(\eta) a_{-\vec{k}} \right] \quad (39)$$

3.3. The two-point function

From the above we arrive at the two-point function evaluated at late times

$$\langle \varphi_{\vec{k}_1} \varphi_{\vec{k}_2} \rangle = (-1)^{m_1} \delta_{-\vec{k}_1, \vec{k}_2} |\varphi_{k_1}^{\text{cl}}(\pi/2)|^2 = (-1)^{m_1} \delta_{-\vec{k}_1, \vec{k}_2} \frac{\dot{\rho}_*^2 \lambda^2}{2\sqrt{k_1^2 - 3(k_1^2 - 4)}}. \quad (40)$$

In order to sum expectation values of φ into expectation values of the curvature perturbation, ζ , one can use the non-linear δN formalism, which will be described in more detail in §5.3 below. Here we merely quote the result for the two-point function in

to a superficially different but gauge-equivalent metric

$$ds^2 = -\tilde{N}^2 d\tilde{t}^2 + e^{2\rho} \gamma_{ij} (d\tilde{x}^i + \tilde{N}^i d\tilde{t}) (d\tilde{x}^j + \tilde{N}^j d\tilde{t}), \quad \tilde{\varphi} \equiv 0. \quad (34)$$

The coordinate transformation which allows us to pass between these two equivalent forms can be written

$$\tilde{t} = t + \frac{\varphi_2}{\dot{\phi}}, \quad \tilde{x}^i = x^i + \frac{\dot{\rho}}{2\dot{\phi}} \varphi_2^i. \quad (35)$$

* The $(-1)^m$ term here comes from $[Y_l^m(\theta, \phi)]^* = (-1)^m Y_l^{-m}(\theta, \phi)$.

order to demonstrate that the power spectrum is consistent with the case of flat spatial slices, giving

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle = (-1)^{m_1} \delta_{-\vec{k}_1, \vec{k}_2} \frac{\dot{\rho}_*^4}{2\dot{\phi}_*^2 k_1^3} [1 + O(k_1^{-2})] . \quad (41)$$

It follows that the power spectrum of fluctuations in ζ is the same as in the \mathbb{R}^3 case [22], up to very small corrections.

3.4. The scalar spectral tilt

In the absence of non-gaussianity, which will be studied in detail in §§4–5, the principal discriminant among models of inflation is the spectral tilt. Although Eq. (41) shows that the *magnitude* of the power spectrum of fluctuations approximately matches the flat space spectrum if k is not small, the tilt contains curvature terms from λ and $\ddot{\rho}_*$. These curvature terms must be taken into account when determining whether the model is compatible with the WMAP constraints on the spectral index n_s .

In the case of flat equal time slices one removes a term $1/k^3$ from the two-point function when determining the tilt. This is because the tilt is a measure of the deviation from scale invariance. Since the degeneracy of modes on S^3 with a given k increases discretely with k , the notion of scale invariance is not natural on S^3 ; however, experiment proceeds on the supposition of flat equal time slices and thereby demands that we compute the tilt in the usual way.

For large k the term $\dot{\rho}_*^{-2} e^{-2\rho_*}$ is small and so the tilt which follows from Eq. (41) is

$$n_s - 1 = 2\eta_* - 6\epsilon_* + O(k^{-2}) . \quad (42)$$

where the slow-roll parameters ϵ and η are defined by[‡]

$$\epsilon \equiv \frac{\dot{\phi}^2}{2\dot{\rho}^2} \quad \text{and} \quad \eta \equiv -\frac{\ddot{\phi}}{\dot{\phi}\dot{\rho}} + \frac{\dot{\phi}^2}{2\dot{\rho}^2} . \quad (43)$$

The last term in Eq. (42) will be significantly smaller than the first two for $k \gtrsim 20$, and so we will recover the usual result for flat spatial slices. In this case it is known observationally that the tilt is small [2], so it follows that the combination of ϵ_* and η_* in Eq. (42) is too: generically, they are both small. More generally, Eq. (42) shows that the two-point function is unlikely to be strongly sensitive to curvature unless good constraints can be obtained on a possible running.

[‡] In the case of flat spatial slices, one often uses $\epsilon_H \equiv -\ddot{\rho}/\dot{\rho}^2$ instead of the quantity ϵ ; here these differ by a curvature term, so that $\epsilon = \epsilon_H + \dot{\rho}^{-2} e^{-2\rho}$. The difference may be significant for those modes passing outside the horizon when there is substantial curvature, so we are taking Eq. (43) to be the fundamental choice because the field is rolling slowly on a Hubble timescale. This also leads to $(V')^2/2V^2 \approx \epsilon/\lambda^4$ and $V''/V \approx \eta$. Note that ϵ_H need not be small for modes passing out of the horizon early on.

3.5. Projecting onto the sky

In order to properly compare results in the S^3 case with those in the \mathbb{R}^3 case one should project onto the sky and compare the resulting *angular* expectation values of ζ . For this we need to consider the S^3 harmonics evaluated at χ_L on the sky, where $\chi_L \ll 1$. From Eq. (37) and the eigenvalue equation $\Delta Q_{\vec{k}} = -(k^2 - 1)Q_{\vec{k}}$ we see that $\Pi_k^l(\chi)$ obeys the defining equation

$$\frac{1}{\sin^2 \chi} \frac{d}{d\chi} \left(\sin^2 \chi \frac{d}{d\chi} \Pi_k^l(\chi) \right) - \frac{l(l+1)}{\sin^2 \chi} \Pi_k^l(\chi) + (k^2 - 1) \Pi_k^l(\chi) = 0. \quad (44)$$

In the limit $\chi \ll 1$, $k \gg 1$ and $l \gg 1$ the solutions become proportional to a spherical Bessel function, $\Pi_k^l(\chi) \propto j_l(k\chi)$. The correct normalization can be obtained by studying Eq. (38) in the limit $\chi \rightarrow 0$ and yields

$$\Pi_k^l(\chi) \approx \sqrt{\frac{2}{\pi}} k j_l(k\chi) \quad (45)$$

for $\chi \ll 1$, $k \gg 1$ and $l \gg 1$. In the limit $\chi_L \ll 1$ the surface of last scattering is becoming close to our point of observation, compared with the radius of curvature of the S^3 . We therefore expect that curvature ceases to play any role and the two-point function projected onto the sky goes over to its flat space counterpart. Indeed, one can show that in this limit (and with the restriction $l \gg 1$ which guarantees we are looking on small angular scales)

$$\langle \zeta_{l_1 m_1} \zeta_{l_2 m_2} \rangle \approx (-1)^{m_1} \delta_{l_1, l_2} \delta_{m_1, -m_2} \int_0^\infty \frac{dk}{k} \frac{\dot{\rho}_*^4}{\phi_*^2} \frac{1}{\pi} \left[j_l(k\chi_L) \right]^2 \quad (46)$$

where the approximation of continuing the integral to 0 is valid because the main contribution comes from the region $k \approx l\chi_L^{-1}$. Eq. (46) is equivalent to the well-known result in the \mathbb{R}^3 case, which demonstrates the consistency of our calculation.

4. The third-order action

In the previous section we expanded the action to second order in the small fluctuation φ , which is sufficient to determine its two-point statistics and therefore the power spectrum. If we wish to go further, however, and determine the leading non-linearity then it is necessary to obtain a description of the process by which three φ quanta can interact. This information is provided by the third order term in the expansion of the action, which we will determine in the present section before going on to study the three-point function in §5.

After expanding S according to its definition, it follows that the third order term can be written

$$\begin{aligned} S_3 = \frac{1}{2} \int dt d^3x e^{3\rho} \sqrt{\gamma} & \left(-\frac{1}{3} V_0''' \varphi^3 - V_0'' \delta N \varphi^2 - \delta N \dot{\varphi}^2 - 2\dot{\varphi} N^i \varphi_{,i} \right. \\ & - \delta N e^{-2\rho} \gamma^{ij} \varphi_{,i} \varphi_{,j} + 6\dot{\rho}^2 \delta N^3 + 4\dot{\rho} \Delta \chi \delta N^2 - \delta N \chi_{[ij} \chi^{ij} \\ & \left. + \delta N (\Delta \chi)^2 + 2\dot{\phi} \delta N N^i \varphi_{,i} - \dot{\phi}^2 \delta N^3 + 2\dot{\phi} \delta N^2 \dot{\varphi} \right), \end{aligned} \quad (47)$$

where to avoid unnecessary clutter we have denoted the covariant derivative on the unit three-sphere by a vertical bar, so that $X_{|i} \equiv \nabla_i X$ and $X^{[i} \equiv h^{ij} X_{|j}$.

From this expression we require the leading order term in slow-roll. This will allow us to compute the expectation value $\langle \varphi \varphi \varphi \rangle$ to the first non-trivial order at horizon crossing, after which we must perform a gauge transformation to determine $\langle \zeta \zeta \zeta \rangle$. After horizon crossing we expect the correlation functions of the curvature perturbation to be approximately conserved.

To identify the leading slow-roll terms, we must relate the derivatives of the potential to the motion of the background field ϕ . For this purpose one can use the relations

$$V_0''' = \left(\frac{6\dot{\rho}}{\dot{\phi}} - 3\frac{\ddot{\phi}}{\dot{\phi}^2} \right) e^{-2\rho} + \mathcal{O}(\epsilon^{3/2}), \quad \text{and} \quad V_0'' = -3e^{-2\rho} + \mathcal{O}(\epsilon). \quad (48)$$

All other terms in Eq. (47) are suppressed by at least one power of $\dot{\phi}/\dot{\rho}$, so the leading contribution to the action comes from V_0''' . On flat spatial slices this term is usually negligible [31, 32], since it is proportional to powers of $\dot{\phi}$ and $\ddot{\phi}$. These terms are indeed present in Eq. (48), but are accompanied by a term of order $\dot{\rho}/\dot{\phi}$ whose source is the curvature term in the background scalar field equation, Eq. (17). Accordingly, a significant three-point interaction can be present at early times when the scalar field is behaving in a way quite different from the flat slicing expectation.

We are assuming that a slow-roll hierarchy exists at the time of evaluation, so it follows that the $\ddot{\phi}$ contribution can be discarded and the leading contribution to the action can be written

$$S_3 = - \int dt d^3x \sqrt{\gamma} e^\rho \frac{\dot{\rho} \varphi^3}{\dot{\phi}}. \quad (49)$$

This is of order $\dot{\rho}/\dot{\phi} \sim \epsilon^{-1/2}$ and is a qualitatively new contribution which is not present in the interactions among φ quanta on \mathbb{R}^3 spatial slices [3]. This term is suppressed by two powers of the scale factor compared to the vacuum energy density and therefore appears in Eq. (48) proportional to $e^{-2\rho}$, which implies that at late times where $e^\rho \rightarrow \infty$ it no longer contributes to the interactions. Thus, for large k we can expect that the effect of primordial curvature disappears and we recover the standard flat space result.

In §5 we will determine the contribution that this interaction makes to the $\langle \zeta \zeta \zeta \rangle$ expectation value. In fact, we will see that it gives rise to a term which behaves like ϵ^{-2} , and which can therefore dominate over those terms which are common to flat hypersurfaces and S^3 hypersurfaces (to be described below), which behave at leading order like ϵ^{-1} . Of course, as we have described, other terms in the slow-roll expansion of V_0''' will dominate if $e^{-2\rho}$ is small—that is, if the curvature of the spatial hypersurfaces is small. We can only expect the term in equation (49) to dominate in the three point function if there has not been much inflation prior to the mode leaving the horizon.

The next-to-leading order terms contribute proportional to $\epsilon^{1/2}$. Their effect in the action can be written

$$S_3 = \frac{1}{2} \int dt d^3x e^{3\rho} \sqrt{\gamma} \left(\frac{\ddot{\phi}}{\dot{\phi}^2} e^{-2\rho} \varphi^3 + 3e^{-2\rho} \delta N \varphi^2 - \delta N \dot{\varphi}^2 - 2\dot{\varphi} N^i \varphi_{|i} - \delta N e^{-2\rho} \varphi^{[i} \varphi_{|i} \right). \quad (50)$$

These terms give rise to contributions which are common to both S^3 and \mathbb{R}^3 , and are suppressed by a power of ϵ compared to the leading curvature term (49). They are the same as the terms obtained by Maldacena [3]. Adding both these contributions together, integrating by parts, and using the background equations for ρ and ϕ , we obtain

$$S_3 = - \int dt d^3x \left\{ \sqrt{\gamma} \left(e^\rho \frac{\dot{\rho}}{\dot{\phi}} \varphi^3 + e^{5\rho} \dot{\phi} \dot{\varphi}^2 (\Delta + 3)^{-1} \dot{\varphi} \right) + \frac{\delta L_2}{\delta \varphi} f(\varphi) \right\} \quad (51)$$

where $f(\varphi)$ is an auxiliary function defined by

$$f(\varphi) \equiv \frac{\dot{\phi}}{8\dot{\rho}} \varphi^2 - \frac{3\dot{\phi}}{8\dot{\rho}} (\Delta + 3)^{-1} (\varphi^2) - \frac{\dot{\phi}}{4\dot{\rho}} (\Delta + 3)^{-1} \{ \varphi (\Delta + 3) \varphi \} + \dots \quad (52)$$

and ‘ \dots ’ represents terms which vanish at late times. The term involving $\delta L_2 / \delta \varphi$ is proportional to the leading-order equation of motion and therefore vanishes when we take the interaction picture field to be on-shell. However, it cannot simply be discarded; it records the contribution of boundary terms which were generated after integrating by parts and which we have not written explicitly [33]. The correct procedure is to make a field redefinition to remove these terms, by introducing a shifted field φ_c , defined by

$$\varphi = \varphi_c + f(\varphi_c). \quad (53)$$

This removes the nuisance terms proportional to the equation of motion *and* the boundary terms, giving a simplified action

$$S_3 = - \int dt d^3x \sqrt{\gamma} \left(e^\rho \frac{\dot{\rho}}{\dot{\phi}} \varphi_c^3 + e^{5\rho} \dot{\phi} \dot{\varphi}_c^2 (\Delta + 3)^{-1} \dot{\varphi}_c \right). \quad (54)$$

We may now compute the simpler correlation functions of φ_c , and rewrite the result in terms of the interesting field φ by using Eq. (53).

5. The three-point function

5.1. The φ correlation function

The final step is to calculate the three-point function in the comoving gauge (7). As we have described, the comoving curvature perturbation ζ is conserved after horizon exit in the absence of non-adiabatic pressure. Once a given scale falls back inside the horizon, ζ can be used to seed the subsequent calculation of temperature and density fluctuations in the coupled baryon–photon plasma.

To obtain the correlation functions of ζ entails a number of steps. The first requires that we obtain the contribution from the reduced third order action (54), and rewrite it in terms of the correlation functions of the original field φ using the field redefinition (52). Once this has been done it is necessary to determine how the correlation functions of ζ are related to those of φ . Fortunately, there is a simple prescription (the so-called “ δN formalism”) which is valid on large scales [34, 35, 27, 4]. The result is that $\zeta(t, \vec{x})$ evaluated at any time t later than the horizon-crossing time t_* can be written

$$\zeta(t, \vec{x}) \equiv \delta N(t, \vec{x}) = \sum_{n=1}^{\infty} \frac{1}{n!} \left\{ \frac{\partial^n}{\partial \phi_*^n} N(t, t_*) \right\} [\varphi_*(t_*, \vec{x})]^n, \quad (55)$$

where t_* is taken to be a spatial slice on which the curvature perturbation ζ vanishes, t labels a slice of uniform energy density, and N is the number of e-folds between these two slices. Note that this formula applies only in coordinate space and must be treated accordingly when transforming to harmonic modes. Eq. (55) can also be understood in terms of a further field redefinition which changes the action from the uniform curvature to comoving gauge [3]. The whole computation is performed with the interaction picture fields using either (31) or (27) according to whether k is large or small.

Let us first determine the contribution from the first term in the reduced third-order action, Eq. (54). Provided we are only interested in tree-level amplitudes, the interaction Hamiltonian is simply given by minus the interaction term in the Lagrangian [36, 37], so that $H_{\text{int } 3} = -L_{\text{int } 3}$. It follows that the contribution from the first term in (54), for large k , is

$$\begin{aligned} \langle \varphi_{\vec{k}_1} \varphi_{\vec{k}_2} \varphi_{\vec{k}_3} \rangle &\supseteq -i \int_{i\infty}^{\pi/2-\varepsilon} d\eta \langle [\varphi_{\vec{k}_1}(\pi/2) \varphi_{\vec{k}_2}(\pi/2) \varphi_{\vec{k}_3}(\pi/2), H_{\text{int } 3}(\eta)] \rangle \\ &= \dots - \frac{6\dot{\rho}_*^5 \lambda^4}{8k_1^2 k_2^2 k_3^2 \dot{\phi}_*} e^{-i\pi k_t/2} \int d^3x \sqrt{\gamma} Q_{\vec{k}_1} Q_{\vec{k}_2} Q_{\vec{k}_3} J + \text{c.c.}, \end{aligned} \quad (56)$$

where “c.c.” denotes the complex conjugate of the preceding expression and $k_t \equiv k_1 + k_2 + k_3$. The function J is defined by

$$J = \int_{i\infty}^{\pi/2-\varepsilon} d\eta \frac{\sin \eta}{\cos^2 \eta} \left(\cos \eta - \frac{i}{k_1} \right) \left(\cos \eta - \frac{i}{k_2} \right) \left(\cos \eta - \frac{i}{k_3} \right) e^{i k_t \eta}. \quad (57)$$

In Eqs. (56)–(57) we have carried the integration over conformal time to within a small parameter ε (not to be confused with the slow-roll parameter ϵ) of future infinity. We can expect that this will be a good approximation because the integral receives its largest contribution from around the time of horizon exit and thereafter does not evolve appreciably, so there is little error in continuing the integration into the infinite future. Inside the horizon the integral is taken over a contour which turns a right-angle at $\eta = 0$ and approaches infinity along the positive imaginary axis, corresponding to evaluation of the interaction in the Hartle–Hawking state. This is equivalent to the interacting vacuum of the full theory.

Eqs. (56)–(57) give rise to a term in the φ correlation function which takes the form

$$\langle \varphi_{\vec{k}_1} \varphi_{\vec{k}_2} \varphi_{\vec{k}_3} \rangle \supseteq -\frac{3\dot{\rho}_*^5}{2\dot{\phi}_* k_1^3 k_2^3 k_3^3} \int d^3x \sqrt{\gamma} Q_{\vec{k}_1} Q_{\vec{k}_2} Q_{\vec{k}_3} \left(-2 \frac{k_1 k_2 k_3}{k_t^2} + k_t - \frac{1}{k_t^2} \sum_{i \neq j} k_i k_j^2 \right) \quad (58)$$

In order to obtain the contribution this makes to the ζ correlation function we will shortly see that the δN formalism tells us we must multiply by $(-\dot{\rho}_*/\dot{\phi}_*)^3$. The result will be proportional to $\epsilon^{-2} k^{-8} \dot{\rho}_*^4$, whereas the usual terms from the calculation with flat constant time hypersurfaces are proportional to $\epsilon^{-1} k^{-6} \dot{\rho}_*^4$. So we expect the terms given in Eq. (58) to dominate for small k .

On the other hand, some of the usual terms in the three point function are obtained by calculating the contribution from the second term in Eq. (54). These terms are given

by

$$\langle \varphi_{\vec{k}_1} \varphi_{\vec{k}_2} \varphi_{\vec{k}_3} \rangle \supseteq -\frac{\dot{\phi}_* \dot{\rho}_*^3}{2k_1^3 k_2^3 k_3^3} \int d^3x \sqrt{\gamma} Q_{\vec{k}_1} Q_{\vec{k}_2} Q_{\vec{k}_3} \sum_{i>j} \frac{k_i^2 k_j^2}{k_t}. \quad (59)$$

Adding on the terms from the field redefinition (see section (5.2)), given by Eqs. (52) and (53), and taking the dominant terms for large k , we obtain

$$\langle \varphi_{\vec{k}_1} \varphi_{\vec{k}_2} \varphi_{\vec{k}_3} \rangle \supseteq -\frac{\dot{\phi}_* \dot{\rho}_*^3}{8k_1^3 k_2^3 k_3^3} \int d^3x \sqrt{\gamma} Q_{\vec{k}_1} Q_{\vec{k}_2} Q_{\vec{k}_3} \left(\frac{4}{k_t} \sum_{i>j} k_i^2 k_j^2 - \frac{1}{2} \sum_i k_i^3 + \frac{1}{2} \sum_{i \neq j} k_i k_j^2 \right) \quad (60)$$

5.2. Terms due to the field redefinition

The field redefinition given by Eqs. (52)–(53) contributes a number of terms to the three point function. These are calculated with the aid of a convolution (see Eq. (63)) and then contraction of the fields in such a way that only connected diagrams are produced. The first term of Eq. (52) results in a calculation much like that which will be done in section §5.3. The second and third terms in Eq. (52) are given by a marginally more complicated calculation where one must integrate by parts to throw the $(\Delta + 3)^{-1}$ onto the $Q_{\vec{k}}$ in the convolution, and then contract the fields. The sum total of the redefinition, Eq. (52), is then

$$\begin{aligned} \langle \varphi_{\vec{k}_1} \varphi_{\vec{k}_2} \varphi_{\vec{k}_3} \rangle - \langle \varphi_{c\vec{k}_1} \varphi_{c\vec{k}_2} \varphi_{c\vec{k}_3} \rangle = \\ \frac{\dot{\phi}_* \dot{\rho}_*^3 \lambda^4}{16 \prod_l k_l^3} \int d^3x \sqrt{\gamma} Q_{\vec{k}_1} Q_{\vec{k}_2} Q_{\vec{k}_3} \left(\sum_i k_i^3 + 3 \sum_i k_i - \sum_{i \neq j} k_i^2 k_j \right) \end{aligned} \quad (61)$$

5.3. The δN formalism

Finally, one uses the δN prescription to obtain the full ζ correlation function. Multiplying three copies of Eq. (55), taking correlations using Wick's theorem and truncating to tree-level terms which do not involve unconstrained momentum integrations, we find

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle = \left(\frac{\delta N}{\delta \phi_*} \right)^3 \langle \varphi_{\vec{k}_1} \varphi_{\vec{k}_2} \varphi_{\vec{k}_3} \rangle + \frac{1}{2} \left(\frac{\delta N}{\delta \phi_*} \right)^2 \frac{\delta^2 N}{\delta \phi_*^2} [\langle \varphi_{\vec{k}_1} \varphi_{\vec{k}_2} (\varphi \star \varphi)_{\vec{k}_3} \rangle + \text{cyclic}], \quad (62)$$

where \star denotes a ‘convolution,’

$$(\varphi \star \varphi)_{\vec{k}}(t) \equiv \int d^3x \sqrt{\gamma} Q_{\vec{k}}^*(\vec{x}) \varphi^2(t, \vec{x}), \quad (63)$$

“cyclic” indicates that all cyclic permutations of $\{1, 2, 3\}$ should be included in the sum, and N measures by how many e-folds the mode in question is outside the horizon (see description below Eq. (55)). Since we are working with a single-field model of inflation, the derivatives of N can be evaluated directly, yielding

$$\frac{\delta N}{\delta \phi_*} = -\frac{\dot{\rho}_*}{\dot{\phi}_*}, \quad \text{and} \quad \frac{\delta^2 N}{\delta \phi_*^2} = \frac{\dot{\rho}_* \ddot{\phi}_*}{\dot{\phi}_*^3} + \frac{1}{2} - \frac{1}{\dot{\phi}_*^2 c_*^{2\rho}}. \quad (64)$$

5.4. The ζ correlation function

Collecting all these terms, the final result for the three point function evaluated on a late time slice is

$$\begin{aligned} \langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle = & \frac{\dot{\rho}_*^6}{8\dot{\phi}_*^2 k_1^3 k_2^3 k_3^3} \int d^3x \sqrt{\gamma} Q_{\vec{k}_1} Q_{\vec{k}_2} Q_{\vec{k}_3} \\ & \times \left[\frac{4}{k_t} \sum_{i>j} k_i^2 k_j^2 + \frac{1}{2} \sum_i k_i^3 + \frac{1}{2} \sum_{i \neq j} k_i k_j^2 + \frac{2\dot{\rho}_* \ddot{\phi}_*}{\dot{\phi}_*^3} \sum_i k_i^3 \right. \\ & \left. + \frac{2\dot{\rho}_*^2}{\dot{\phi}_*^2} \left(-12 \frac{k_1 k_2 k_3}{k_t^2} + 6k_t - \frac{6}{k_t^2} \sum_{i \neq j} k_i k_j^2 - \frac{1}{\dot{\rho}_*^2 e^{2\rho_*}} \sum_i k_i^3 \right) \right] \quad (65) \end{aligned}$$

Of course, this uses the large k approximation for φ^{cl} and so it can only be regarded as approximate.

The leading term in this expression scales with momentum like $[k^{-6}]$. This is the term computed by Maldacena, and is dominant for sufficiently large k , that is, on small scales. The correction terms we have computed scale with momentum like $[k^{-8}]$. These terms can become dominant on larger scales. On sufficiently large scales, of course, one must remember that Eq. (65) will be accompanied by other corrections that scale even faster as powers of momentum, like $[k^{-2n}]$ for any integer $n \geq 3$, and that these corrections will eventually overwhelm the ones we have computed. In this regime, for accuracy, one should use the full expression for the modes (27) and perform a numerical assessment of the three point function.

5.5. An estimate of f_{NL}

We are finally in a position to estimate the magnitude of the observable non-linearity parameter f_{NL} which is produced by the sensitivity to curvature in Eq. (65). We define the sign of f_{NL} according to the WMAP convention, where the bispectrum is parameterized in harmonic space via

$$B(k_1, k_2, k_3) = \frac{6}{5} f_{\text{NL}} \{P(k_1)P(k_2) + P(k_1)P(k_3) + P(k_2)P(k_3)\}. \quad (66)$$

One must be careful in interpreting this formula for modes on S_3 , because when we work with the harmonics $Q_{\vec{k}}$, the k_i no longer have the same meaning as in flat space. The correct way to compare the magnitude of f_{NL} between models with different spatial geometries is to project onto the sky and compare the non-linearity parameter in the angular expectation values of ζ .

To obtain the correct projection, we begin by writing the three- ζ correlator in terms of a function $\xi(k_1, k_2, k_3)$ defined by

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle = \int d^3x \sqrt{\gamma} Q_{\vec{k}_1} Q_{\vec{k}_2} Q_{\vec{k}_3} \xi(k_1, k_2, k_3). \quad (67)$$

To obtain the angular correlation function of the ζ s, we follow the route outlined in §3.5: the correlator in harmonic space is transformed back to coordinate space, and evaluated

at the radial distance corresponding to last scattering. In terms of the polar coordinate on S^3 , this is given by $\chi = \chi_L$. One finds

$$\begin{aligned} \langle \zeta_{l_1 m_1} \zeta_{l_2 m_2} \zeta_{l_3 m_3} \rangle_{\chi_L} &= C_{l_1 l_2 l_3}^{m_1 m_2 m_3} \sum_{k_i \geq (l_i+1)} \Pi_{k_1}^{l_1}(\chi_L) \Pi_{k_2}^{l_2}(\chi_L) \Pi_{k_3}^{l_3}(\chi_L) \\ &\times \int_0^\pi \sin^2 \chi d\chi \Pi_{k_1}^{l_1}(\chi) \Pi_{k_2}^{l_2}(\chi) \Pi_{k_3}^{l_3}(\chi) \xi(k_1, k_2, k_3) \end{aligned} \quad (68)$$

where

$$C_{l_1 l_2 l_3}^{m_1 m_2 m_3} = \int d\Omega^2(\hat{x}) Y_{l_1}^{m_1}(\hat{x}) Y_{l_2}^{m_2}(\hat{x}) Y_{l_3}^{m_3}(\hat{x}), \quad (69)$$

and $d\Omega^2(\hat{x})$ is an element of solid angle on S^2 in the direction of the unit vector \hat{x} . The integration over χ is symmetric or antisymmetric about $\chi = \pi/2$, with the two regions interfering constructively for k_t odd and destructively for k_t even. Moreover, the $\Pi_k^l(\chi)$ under the integral can be replaced by spherical Bessel functions for the range of χ , k and l of interest. It follows that

$$\begin{aligned} \langle \zeta_{l_1 m_1} \zeta_{l_2 m_2} \zeta_{l_3 m_3} \rangle_{\chi_L} &\approx C_{l_1 l_2 l_3}^{m_1 m_2 m_3} \left(\frac{2}{\pi}\right)^3 \int_0^{\pi/2} \sin^2 \chi d\chi \\ &\times \left\{ \sum_{k_1 \geq (l_1+1)} k_1^2 j_{l_1}(k_1 \chi) j_{l_1}(k_1 \chi_L) \right\} \left\{ 1 \leftrightarrow 2 \right\} \left\{ 1 \leftrightarrow 3 \right\} \\ &\times [1 - (-1)^{k_t}] \xi(k_1, k_2, k_3) \end{aligned} \quad (70)$$

The influence of the $[1 - (-1)^{k_t}]$ term averages out to 1 as we sum over, say, k_3 and the summation over the k_i may be replaced by integrals to give

$$\begin{aligned} \langle \zeta_{l_1 m_1} \zeta_{l_2 m_2} \zeta_{l_3 m_3} \rangle_{\chi_L} &\approx C_{l_1 l_2 l_3}^{m_1 m_2 m_3} \left(\frac{2}{\pi}\right)^3 \int_0^{\pi/2} \sin^2 \chi d\chi \\ &\times \left\{ \int_{(l_1+1)}^\infty dk_1 k_1^2 j_{l_1}(k_1 \chi) j_{l_1}(k_1 \chi_L) \right\} \left\{ 1 \leftrightarrow 2 \right\} \left\{ 1 \leftrightarrow 3 \right\} \xi(k_1, k_2, k_3) \end{aligned} \quad (71)$$

First consider the part of ξ which scales with momentum like $[k^{-6}]$, coming from those terms in Eq. (65) which were computed by Maldacena and are present in the flat space expectation value. In order to carry out the integrations in Eq. (71) explicitly it is most convenient if the terms involving each k_i in ξ factorize, so that each k_i integral can be evaluated independently. The necessary components of ξ are

$$\xi(k_1, k_2, k_3) = \frac{\dot{\rho}_*^6}{8\dot{\phi}_*^2 k_1^3 k_2^3 k_3^3} \left(\frac{4}{k_t} \sum_{i>j} k_i^2 k_j^2 + \frac{1}{2} \sum_i k_i^3 + \frac{1}{2} \sum_{i \neq j} k_i k_j^2 + \frac{2\dot{\rho}_* \ddot{\phi}_*}{\dot{\phi}_*^3} \sum_i k_i^3 \right) + \dots \quad (72)$$

There is a small variation with k owing to the variation of $\dot{\rho}_*$, $\dot{\phi}_*$ and $\ddot{\phi}_*$ with the epoch of horizon exit. Ignoring this slow variation, however, the only term which does not factorize involves $1/k_t$. A similar obstruction was encountered by Smith & Zaldarriaga [38], who found a factorizable form by introducing a Schwinger parameter,

$$\frac{1}{k_t} \equiv \int_0^\infty dt e^{-t(k_1+k_2+k_3)}. \quad (73)$$

To understand which regions make significant contributions to the integral in Eq. (71) one can account for the influence of the Bessel functions using a stationary phase method. In practice, one employs a WKB approximation to write these functions as an amplitude multiplied by a term with rapidly varying phase. An appropriate WKB solution can be constructed from the defining equation of the spherical Bessel functions,

$$\frac{d^2}{dz^2} [z j_l(z)] = \left(\frac{l(l+1)}{z^2} - 1 \right) [z j_l(z)]. \quad (74)$$

The derivative of the WKB phase satisfies

$$\frac{d}{dk}(\text{WKB phase}) = \frac{d}{dk} \int_{\sqrt{l(l+1)}}^{k\chi} \sqrt{1 - \frac{l(l+1)}{z^2}} dz = \chi \sqrt{1 - \frac{l(l+1)}{k^2 \chi^2}}. \quad (75)$$

First consider contributions to the χ integral in Eq. (71) from the region where $\chi \gg \chi_L$. There are two cases, depending whether k^2 is larger or smaller than $l(l+1)/\chi^2$. For very large k , the stationary phase approximation implies that there is essentially no contribution to the integral. We are therefore left with the region $k^2 \leq l(l+1)/\chi^2$. Since $\chi \gg \chi_L$ it follows that $k^2 \chi_L^2 \ll l(l+1)$, and therefore $j_l(k\chi_L)$ is becoming exponentially suppressed. It follows that there is also a negligible contribution in this case. The only significant contribution comes from the region where $\chi \lesssim \chi_L$. In this region we can approximate $\sin \chi \approx \chi$ because $\chi_L \ll 1$; the error this induces in the region $\chi > \chi_L$ is immaterial because the integrand is exponentially suppressed there. Moreover, in view of what has been said about the region where $\chi \gg \chi_L$, there is very little error in extending the range of χ integration to infinity. Accordingly, we have

$$\begin{aligned} \langle \zeta_{l_1 m_1} \zeta_{l_2 m_2} \zeta_{l_3 m_3} \rangle_{\chi_L} &\approx C_{l_1 l_2 l_3}^{m_1 m_2 m_3} \left(\frac{2}{\pi} \right)^3 \int_0^\infty \chi^2 d\chi \\ &\times \left\{ \int_{l_1+1}^\infty dk_1 k_1^2 j_{l_1}(k_1 \chi) j_{l_1}(k_1 \chi_L) \right\} \left\{ 1 \leftrightarrow 2 \right\} \left\{ 1 \leftrightarrow 3 \right\} \xi(k_1, k_2, k_3). \end{aligned} \quad (76)$$

We are assuming that ξ is factorizable, but if it is not a similar discussion applies after the introduction of Schwinger parameters.

Let us relate this expression to the flat space limit, where $\chi_L \ll 1$. There is only an exponentially suppressed contribution to each k integral from the region $k_i \in [0, l_i + 1]$, because if $\chi_L \ll 1$ then it must also be true that $k_i^2 \chi_L^2 \ll l_i(l_i + 1)$ for each i , and it follows that the Bessel function $j_{l_i}(k_i \chi_L)$ is undergoing exponential suppression in this region. We therefore incur essentially no penalty in extending the lower limit of integration to 0, giving

$$\begin{aligned} \langle \zeta_{l_1 m_1} \zeta_{l_2 m_2} \zeta_{l_3 m_3} \rangle_{\chi_L} &\approx C_{l_1 l_2 l_3}^{m_1 m_2 m_3} \left(\frac{2}{\pi} \right)^3 \int_0^\infty \chi^2 d\chi \\ &\times \left\{ \int_0^\infty dk_1 k_1^2 j_{l_1}(k_1 \chi) j_{l_1}(k_1 \chi_L) \right\} \left\{ 1 \leftrightarrow 2 \right\} \left\{ 1 \leftrightarrow 3 \right\} \xi(k_1, k_2, k_3) \end{aligned} \quad (77)$$

This is the standard formula for the angular bispectrum of the curvature perturbation with flat spatial slices. It follows that in the approximate flat space limit, we can compare

ξ between the cases of \mathbb{R}^3 and S^3 spatial slices to find an estimate of the comparative magnitude of f_{NL} . One finds

$$f_{\text{NL}} = -\frac{5}{12 \sum_l k_l^3} \left[\frac{\dot{\phi}_*^2}{\dot{\rho}_*^2} \left(\frac{4}{k_t} \sum_{i>j} k_i^2 k_j^2 + \frac{1}{2} \sum_i k_i^3 + \frac{1}{2} \sum_{i \neq j} k_i k_j^2 + \frac{2\dot{\rho}_* \ddot{\phi}_*}{\dot{\phi}_*^3} \sum_i k_i^3 \right) \right. \\ \left. + 2 \left(-12 \frac{k_1 k_2 k_3}{k_t^2} + 6k_t - \frac{6}{k_t^2} \sum_{i \neq j} k_i k_j^2 - \frac{1}{\dot{\rho}_*^2 e^{2\rho_*}} \sum_i k_i^3 \right) \right]. \quad (78)$$

The correction terms can be larger than the flat space result when $k \lesssim \epsilon^{-1/2}$.

6. Conclusions

We work in the scenario where the Universe is a slowly rolling but positively curved spacetime formed from a background with constant density spatial slices which are copies of S^3 , as opposed to \mathbb{R}^3 . This possibility is compatible with observation if $|\Omega_k| \lesssim 10^{-2} - 10^{-3}$, depending whether the universe is taken to be positively or negatively curved. Indeed, interpreting present observational limits literally, the diameter of the CMB $d_{\ell=2}$ satisfies approximately $d_{\ell=2} \approx 0.46$ (in units where the radius of curvature is unity), so that the S^2 corresponding to the surface of last scattering spans about a twelfth of the circumference of a great circle on the S^3 of constant time. On such a last scattering wall one can expect that the fluctuation eigenmodes on S^3 of $k \gtrsim 8$ will contribute to the CMB. Modes of long wavelength, corresponding to small k , will pass outside the horizon early and therefore will be sensitive to the presence of curvature. The question is to what degree a realistic observable such as the power spectrum, or f_{NL} , can constrain the appearance of primordial curvature. We have shown that such modes, of small k , will not contribute significantly to the spectrum, but may contribute to f_{NL} .

We perform the appropriate selection of the vacuum for our slowly rolling de Sitter type spacetime, corresponding to the Hartle–Hawking state, and demonstrate how one should execute the calculation of both the two and three point functions on this manifold. Along the way we find exact expressions for the scalar fluctuation modes in global de Sitter coordinates, and since these are rather unwieldy we explore appropriate approximations. For a generic potential V one finds that the contributions to the action due to curvature at leading order in slow roll are remarkably simple. Further, we show how the S^3 slicing can be reliably compared with the conventional flat-slicing calculation on an S^2 surface corresponding to last scattering. In this way we can establish continuity with the \mathbb{R}^3 calculation as one approaches the flat-space limit.

We estimate the contribution of small- k harmonics to the bispectrum. To compare with observations, it is most convenient to state the result in terms of f_{NL} , given in Eq. (78). As written, this expression should be understood to be valid in the approximately equilateral case where all k_i have an approximately equal magnitude. However, we believe our calculation could be generalized, using methods similar to those of Maldacena [3], to accommodate the squeezed limit.

How large an f_{NL} can be obtained? The $k = 1$ mode is the homogeneous background, and $k = 2$ is pure gauge. Therefore the interesting contributions to microwave background fluctuations must come from modes where $k \geq 3$. To estimate an upper limit, we specialize to the equilateral limit where $k_i = k$ for all i . After collecting terms in Eq. (78), this yields

$$f_{\text{NL}} = -\frac{5}{36}(23\epsilon_* - 6\eta_*) - \frac{145}{54k^2}. \quad (79)$$

The first term, proportional to the slow-roll parameters ϵ and η , is precisely the flat-space contribution first derived by Maldacena. The second term is new, and accounts for the leading effect of curvature. It is irrelevant in the limit $k \rightarrow \infty$, but can be large for small k . Since the signal is maximized by choosing k as small as possible, we can obtain an approximate upper bound by setting $k = 3$, leading to $f_{\text{NL}} \sim 0.3$. It follows that the effect of curvature is marginally below the expected detection threshold for the *Planck* satellite, usually supposed to be of order $f_{\text{NL}} \sim 5$ [39], but might lie on the limit of what is practicable with futuristic technology such as a high-redshift 21cm survey [40, 41]. In the latter case, however, one would need some way to distinguish this small signal from the ubiquitous non-linearities of gravity itself [16].

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References

- [1] **WMAP** Collaboration, J. Dunkley *et al.*, *Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Likelihoods and Parameters from the WMAP data*, [arXiv:0803.0586](#).
- [2] **WMAP** Collaboration, E. Komatsu *et al.*, *Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation*, [arXiv:0803.0547](#).
- [3] J. M. Maldacena, *Non-Gaussian features of primordial fluctuations in single field inflationary models*, *JHEP* **05** (2003) 013, [[arXiv:astro-ph/0210603](#)].
- [4] D. H. Lyth and Y. Rodríguez, *The inflationary prediction for primordial non-gaussianity*, *Phys. Rev. Lett.* **95** (2005) 121302, [[arXiv:astro-ph/0504045](#)].
- [5] D. H. Lyth, *Generating the curvature perturbation at the end of inflation*, *JCAP* **0511** (2005) 006, [[arXiv:astro-ph/0510443](#)].
- [6] L. Alabidi and D. Lyth, *Curvature perturbation from symmetry breaking the end of inflation*, *JCAP* **0608** (2006) 006, [[arXiv:astro-ph/0604569](#)].
- [7] M. Sasaki, *Multi-brid inflation and non-Gaussianity*, *Prog. Theor. Phys.* **120** (2008) 159–174, [[arXiv:0805.0974](#)].
- [8] A. Naruko and M. Sasaki, *Large non-Gaussianity from multi-brid inflation*, *Prog. Theor. Phys.* **121** (2009) 193–210, [[arXiv:0807.0180](#)].
- [9] C. T. Byrnes, K.-Y. Choi, and L. M. H. Hall, *Conditions for large non-Gaussianity in two-field slow-roll inflation*, *JCAP* **0810** (2008) 008, [[arXiv:0807.1101](#)].
- [10] C. T. Byrnes, K.-Y. Choi, and L. M. H. Hall, *Large non-Gaussianity from two-component hybrid inflation*, *JCAP* **0902** (2009) 017, [[arXiv:0812.0807](#)].
- [11] N. Bartolo, S. Matarrese, and A. Riotto, *Enhancement of non-Gaussianity after inflation*, *JHEP* **04** (2004) 006, [[arXiv:astro-ph/0308088](#)].

- [12] N. Bartolo, S. Matarrese, and A. Riotto, *CMB Anisotropies at Second Order I*, *JCAP* **0606** (2006) 024, [arXiv:astro-ph/0604416].
- [13] N. Bartolo, S. Matarrese, and A. Riotto, *CMB Anisotropies at Second-Order II: Analytical Approach*, *JCAP* **0701** (2007) 019, [arXiv:astro-ph/0610110].
- [14] C. Pitrou, J.-P. Uzan, and F. Bernardeau, *Cosmic microwave background bispectrum on small angular scales*, arXiv:0807.0341.
- [15] N. Bartolo and A. Riotto, *On the non-Gaussianity from Recombination*, *JCAP* **0903** (2009) 017, [arXiv:0811.4584].
- [16] L. Boubekur, P. Creminelli, G. D’Amico, J. Norena, and F. Vernizzi, *Sachs-Wolfe at second order: the CMB bispectrum on large angular scales*, arXiv:0906.0980.
- [17] R. W. Wald, *Asymptotic behavior of homogeneous cosmological models in the presence of a positive cosmological constant*, *Phys. Rev.* **D28** (1983) 2118–2120.
- [18] X. Chen, R. Easther, and E. A. Lim, *Large non-Gaussianities in single field inflation*, *JCAP* **0706** (2007) 023, [arXiv:astro-ph/0611645].
- [19] T. S. Bunch and P. C. W. Davies, *Quantum Field Theory in de Sitter Space: Renormalization by Point Splitting*, *Proc. Roy. Soc. Lond.* **A360** (1978) 117–134.
- [20] R. Holman and A. J. Tolley, *Enhanced Non-Gaussianity from Excited Initial States*, *JCAP* **0805** (2008) 001, [arXiv:0710.1302].
- [21] P. D. Meerburg, J. P. van der Schaar, and P. S. Corasaniti, *Signatures of Initial State Modifications on Bispectrum Statistics*, arXiv:0901.4044.
- [22] J. J. Halliwell and S. W. Hawking, *The Origin of Structure in the Universe*, *Phys. Rev.* **D31** (1985) 1777.
- [23] B. Freivogel, M. Kleban, M. Rodriguez Martinez, and L. Susskind, *Observational consequences of a landscape*, *JHEP* **03** (2006) 039, [arXiv:hep-th/0505232].
- [24] A. P. S. Yadav and B. D. Wandelt, *Evidence of Primordial Non-Gaussianity (f_{NL}) in the Wilkinson Microwave Anisotropy Probe 3-Year Data at 2.8σ* , *Phys. Rev. Lett.* **100** (2008) 181301, [arXiv:0712.1148].
- [25] R. L. Arnowitt, S. Deser, and C. W. Misner, *Canonical variables for general relativity*, *Phys. Rev.* **117** (1960) 1595–1602.
- [26] H. Kodama and M. Sasaki, *Cosmological Perturbation Theory*, *Prog. Theor. Phys. Suppl.* **78** (1984) 1–166.
- [27] D. H. Lyth, K. A. Malik, and M. Sasaki, *A general proof of the conservation of the curvature perturbation*, *JCAP* **0505** (2005) 004, [arXiv:astro-ph/0411220].
- [28] X. Chen, M.-x. Huang, S. Kachru, and G. Shiu, *Observational signatures and non-Gaussianities of general single field inflation*, *JCAP* **0701** (2007) 002, [arXiv:hep-th/0605045].
- [29] N. D. Birrell and P. C. W. Davies, *Quantum fields in curved space*. Cambridge, UK, 1982. 340pp.
- [30] S. W. Hawking and I. G. Moss, *Fluctuations in the inflationary universe*, *Nucl. Phys.* **B224** (1983) 180.
- [31] T. Falk, R. Rangarajan, and M. Srednicki, *The Angular dependence of the three point correlation function of the cosmic microwave background radiation as predicted by inflationary cosmologies*, *Astrophys. J.* **403** (1993) L1, [arXiv:astro-ph/9208001].
- [32] D. Seery, K. A. Malik, and D. H. Lyth, *Non-gaussianity of inflationary field perturbations from the field equation*, *JCAP* **0803** (2008) 014, [arXiv:0802.0588].
- [33] D. Seery and J. E. Lidsey, *Non-Gaussian inflationary perturbations from the dS/CFT correspondence*, *JCAP* **0606** (2006) 001, [arXiv:astro-ph/0604209].
- [34] A. A. Starobinsky, *Multicomponent de Sitter (inflationary) stages and the generation of perturbations*, *JETP Lett.* **42** (1985) 152–155.
- [35] M. Sasaki and E. D. Stewart, *A general analytic formula for the spectral index of the density perturbations produced during inflation*, *Prog. Theor. Phys.* **95** (1996) 71–78, [arXiv:astro-ph/9507001].

- [36] D. Seery, *One-loop corrections to a scalar field during inflation*, *JCAP* **0711** (2007) 025, [arXiv:0707.3377].
- [37] E. Dimastrogiovanni and N. Bartolo, *One-loop graviton corrections to the curvature perturbation from inflation*, arXiv:0807.2790.
- [38] K. M. Smith and M. Zaldarriaga, *Algorithms for bispectra: forecasting, optimal analysis, and simulation*, arXiv:astro-ph/0612571.
- [39] E. Komatsu and D. N. Spergel, *Acoustic signatures in the primary microwave background bispectrum*, *Phys. Rev.* **D63** (2001) 063002, [arXiv:astro-ph/0005036].
- [40] A. Cooray, *21-cm Background Anisotropies Can Discern Primordial Non- Gaussianity from Slow-Roll Inflation*, *Phys. Rev. Lett.* **97** (2006) 261301, [arXiv:astro-ph/0610257].
- [41] A. Pillepich, C. Porciani, and S. Matarrese, *The bispectrum of redshifted 21-cm fluctuations from the dark ages*, *Astrophys. J.* **662** (2007) 1–14, [arXiv:astro-ph/0611126].